

## Supplementary Material

**Interplay of inhibition and multiplexing : largest eigenvalue statistics**

Saptarshi Ghosh, Sanjiv K. Dwivedi, Mikhail V. Ivanchenko, Sarika Jalan\*

\* E-mail: sarikajalan9@gmail.com

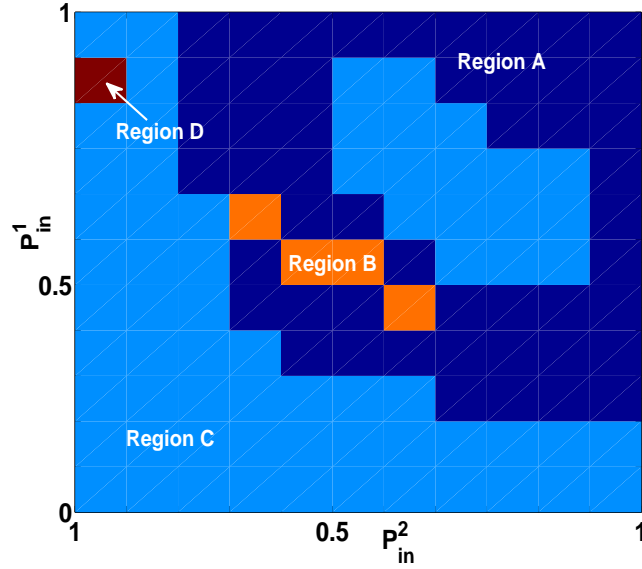


Figure 1: (color online) Phase diagram depicting shape parameter  $\xi$  for accepted GEV distribution for ER-ER multiplex network as a function of IC inclusion probabilities ( $p_{in}$ ) in both the layers. Region B corresponds to the Weibull. Region A stands for undefined distributions. Size of the network  $N=100$  in each layer.

Table 1: Estimated parameters of KS test for fitting GEV and normal distributions of  $R_{max}$  for different network sizes of SF network over a average of 5000 random realization. Other parameters are inhibition inclusion probability  $p_{in} = 0.5$  and average degree  $\langle k \rangle = 6$ .

$N$	$\xi$ of GEV	$\sigma$ of GEV	$\mu$ of GEV	p-value of KS test for GEV	$\mu$ of Normal	$\sigma$ of Normal	p-value of KS test for Normal
100	0.0598	0.4807	2.7906	0.9688	3.0983	0.6685	0.0000
500	0.0367	0.4727	3.5745	0.9223	3.8655	0.6349	0.0000
1000	0.0323	0.4737	3.8953	0.4531	4.1849	0.6303	0.0000
2000	0.0247	0.4671	4.1003	0.192	4.3825	0.6126	0.0000
4000	0.0409	0.5164	4.5638	0.2938	4.8846	0.6893	0.0000

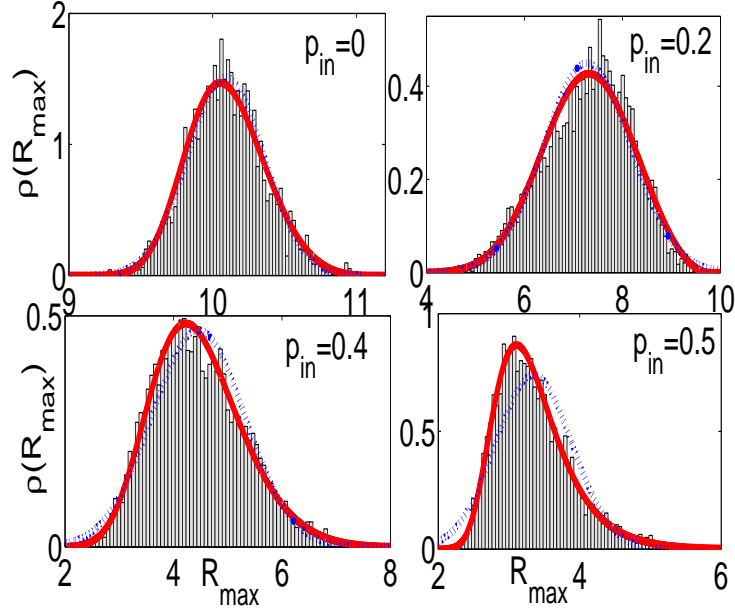


Figure 2: (Color online) Distribution of  $R_{max}$  of SF networks with average degree  $\langle k \rangle = 4$  for various IC inclusion probabilities ( $p_{in}$ ). Histogram is fitted with normal (blue dotted line) and GEV (red solid line) distributions. Network size  $N=500$ .

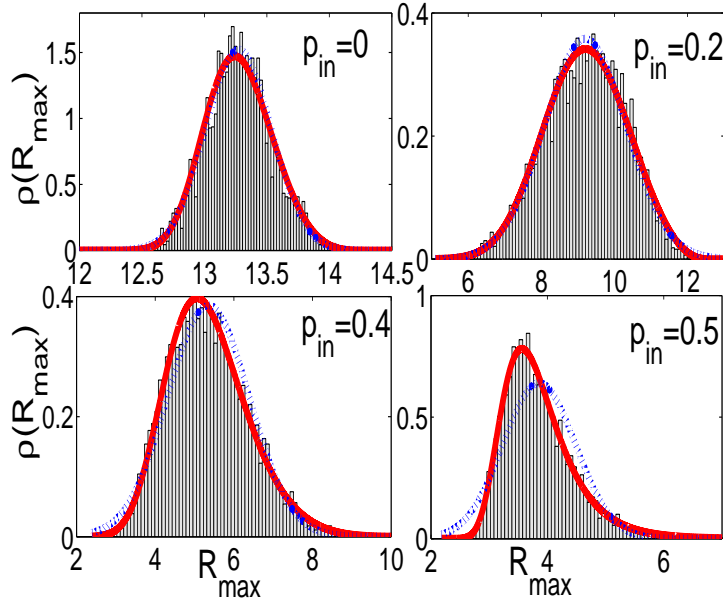


Figure 3: (Color online) Distribution of  $R_{max}$  of SF networks with average degree  $\langle k \rangle = 6$  for various IC inclusion probabilities ( $p_{in}$ ). Histogram is fitted with normal (blue dotted line) and GEV (red solid line) distributions. Network size  $N=500$ .

# Interplay of inhibition and multiplexing : Largest eigenvalue statistics

SAPTARSHI GHOSH<sup>1</sup>, SANJIV K. DWIVEDI<sup>1</sup>, MIKHAIL V. IVANCHENKO<sup>2</sup> AND SARIKA JALAN<sup>1(a)</sup>

1. *Complex Systems Lab, Discipline of Physics, Indian Institute of Technology Indore, Simrol, Indore-453552, India*
2. *Department of Applied Mathematics, Lobachevsky State University of Nizhny Novgorod, Russia*

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**Abstract** –The largest eigenvalue of a network provides understanding to various dynamical as well as stability properties of the underlying system. We investigate interplay of inhibition and multiplexing on the largest eigenvalue statistics of networks. Using numerical experiments, we demonstrate that presence of the inhibitory coupling may lead to a very different behaviour of the largest eigenvalue statistics of multiplex networks than those of the isolated networks depending upon network architecture of the individual layer. We demonstrate that there is a transition from the Weibull to the Gumbel or to the Fréchet distribution as networks are multiplexed. Furthermore, for denser networks, there is a convergence to the Gumbel distribution as network size increases indicating higher stability of larger systems.

*Introduction:* In recent years, network science has attracted researchers from diverse communities owing to its remarkable applicability to understand behavior of many real world complex systems [1]. One of the widely investigated areas in the network science is understanding relation of spectral properties of the network adjacency matrices with dynamical and structural properties of corresponding systems. Particularly, the largest eigenvalue of a network adjacency or coupling matrix has been shown to strongly influence dynamical evolution on the corresponding network, for instance, synchronization properties of coupled Kuramoto oscillators [2] and dynamical properties of neural network [3]. Further, stability of ecological systems are demonstrated to be related with the largest eigenvalue of the interaction matrix of different species [4].

Furthermore, it has been increasingly realized that multiplex networks provide a better framework to investigate structural and dynamical properties of many real world complex systems [5]. As defined in [6], a multiplex network consists of different layers with one to one correlation between the mirror nodes of different layers. Few examples of complex systems, which can be represented in multiplex network framework, are banks, transportation, stock market, etc [7]. A node in the multiplex network architecture may have a very different dynamical behavior

due to the influence of other layers than it has in a single layer network [6]. Further, inhibition in the coupling is known to play a crucial role in functioning and evolution of many real-world networks including brain and ecological networks [8]. We introduce inhibitory and excitatory coupling in different layers of the multiplex networks and investigate statistical properties of  $R_{max}$ . In this Letter, we analyze interplay of multiplexing and inhibition on behavior of  $R_{max}$  of an ensemble of networks coupling matrices under GEV framework. The GEV statistics of independent, identically distributed random variable has been successfully applied to many real world systems including stock markets, natural disasters, galaxy distributions as a model for extreme events [9].

One such system having multiplex network architecture as well as inhibition in the coupling is ecological system, which for instance, may consist of different geographical regions represented as different layers. These layers possess inhibitory and excitatory coupling due to presence of predator-prey, mutualistic and competitive relations among the species. Further, species at different geographical regions may interact with each other due to, for instance, migration.  $R_{max}$  statistics indicates stability of such ecosystems, for example, against extinction of a species. We investigate impact of inhibition on  $R_{max}$  behaviour of multiplex networks and compare them with

<sup>(a)</sup> Corresponding Author: sarikajalan9@gmail.com

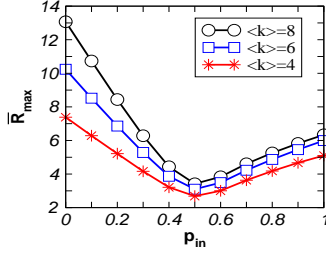


Fig. 1: (Color online) Average  $R_{max}$  as a function of inhibition probability ( $p_{in}$ ) for SF networks of various average degrees  $\langle k \rangle$ . Size of the network remains  $N=100$ . Average is taken over 5000 random realizations of the network.

those of the isolated networks. Since, different layers in a multiplex network can have different network architectures, we explore impact of mixing of different types of network architectures on  $R_{max}$  statistics.

*Theoretical framework:* The adjacency matrix  $A$  of a network has entries  $A_{ij} = 1$  or 0 depending upon whether  $i$  and  $j$  nodes are connected or not. The diagonal entries of  $A$  are zero depicting no self connection. We use the Erdős-Rényi (ER) model to generate a random network [1]. Further, we use configuration model [10] to generate scale-free (SF) networks. To begin with, we consider a network comprising all the excitatory connections leading to a symmetric adjacency matrix i.e.  $A_{ij} = A_{ji} = 1$  having only 0 and 1 matrix elements. An introduction of the inhibitory nodes, with probability  $p_{in}$ , replaces all 1 entries to  $-1$  in the corresponding rows of the adjacency matrix and consequently symmetric property of the matrix is lost [11] as a row corresponding to a inhibitory node will have all  $-1$  non-zero entries, whereas corresponding column may consist of elements having different signs depending upon if they belong to the inhibitory or excitatory nodes. We henceforth declare the adjacency matrix with the directional signs as coupling matrix [12] and investigate distribution of ( $R_{max}$ ) of the ensemble of these matrices having  $-1, 0, 1$  entries.

Further, for the multiplex network, coupling matrix  $A$  can be defined as,

$$A = \begin{pmatrix} A^{(1)} & I \\ I & A^{(2)} \end{pmatrix}, \quad (1)$$

where  $A^{(1)}$  ( $A^{(2)}$ ) represents coupling matrix of the first (second) layer and  $I$  is an unit  $N \times N$  matrix where  $N$  is the size of  $A^{(1)}$  and  $A^{(2)}$  networks. We start with  $P$  random realizations of the multiplex network with ER-ER, SF-SF and ER-SF network topologies and introduce inhibitory couplings in each layer with probability  $p_{in}^{(1)}$  and  $p_{in}^{(2)}$ , respectively and investigate the effect of inhibition in individual layers on the collective  $R_{max}$  behavior of the multiplex network.

GEV distributions can be characterized entirely in terms of three universal probability distribution functions (PDF)

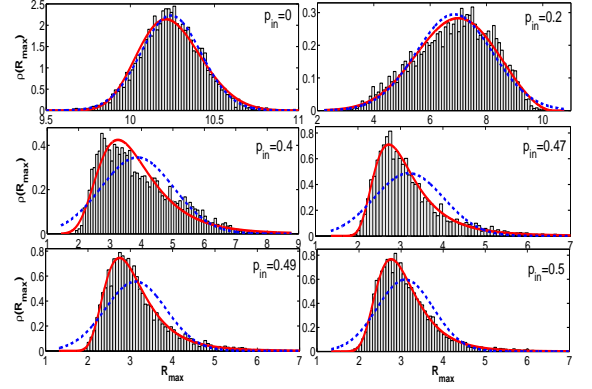


Fig. 2: (Color online) Distribution of  $R_{max}$  for SF networks with  $\langle k \rangle = 6$  and for various inhibitory probabilities ( $p_{in}$ ). Histogram is fitted with normal (blue dotted line) and GEV (red solid line) distributions.

namely Weibull, Gumbel and Fréchet depending on the tail of density function being power law, faster than power law and bounded or unbounded, respectively [9]. The probability density function for three GEV distributions can be written as

$$\rho(x) = \begin{cases} \frac{1}{\sigma} [1 + (\xi \frac{x-\mu}{\sigma})]^{-1-\frac{1}{\xi}} \exp \left[ - (1 + (\xi \frac{x-\mu}{\sigma}))^{-\frac{1}{\xi}} \right] & \text{if } \xi \neq 0 \\ \frac{1}{\sigma} \exp \left( - \frac{x-\mu}{\sigma} \right) \exp \left[ - \exp \left( - \frac{x-\mu}{\sigma} \right) \right] & \text{if } \xi = 0. \end{cases} \quad (2)$$

where  $\mu$ ,  $\sigma$ ,  $\xi$  represent location parameter, scale parameter and shape parameter, respectively. The underlying statistics can be determined by the value of the shape parameter as follows;  $\xi > 0$  (Fréchet Statistics),  $\xi = 0$  (Gumbel statistics) and  $\xi < 0$  (Weibull statistics). We use Kolmogorov-Smirnov (KS) test [13] to characterize the distributions in terms of GEV statistics derived from the numerical simulations.

*Largest eigenvalue statistics for isolated network:* We start the investigation for the isolated ER and SF networks followed by the multiplex network having different network topologies representing each layer. First, we discuss the average behavior of  $R_{max}$  for the isolated SF networks. As inhibitory couplings are introduced in the network, thereby leading to asymmetry in the coupling matrix, spectra of network might start taking up complex eigenvalues. Consequently, the average value of  $R_{max}$  decreases up to  $p_{in} \leq 0.5$  (Fig.1). At  $p_{in} = 0.5$ ,  $\bar{R}_{max}$  shows the minimum value by displaying the global minima. The smallest eigenvalue of a network adjacency matrix ( $\lambda_{min}$ ) with all the positive entries becomes the largest eigenvalue ( $\lambda_{max}$ ) of the same matrix with all the negative entries ( $p_{in} = 1$ ) and as  $\lambda_{max} \neq \lambda_{min}$  for the SF network with all the positive entries. Thus the behaviour of  $R_{max}$  becomes asymmetric about the minima at  $p_{in} = 0.5$  (Fig.1). A similar behavior is observed for different average degrees of the SF networks (Fig. 1).

Next, we probe for the fluctuations in the largest eigen-

value around the mean value. The statistics of  $R_{max}$  is fitted with the normal and GEV distribution (Eq.2). For  $p_{in} = 0$ , we find that the distribution can be modeled using the GEV statistics and value of the shape parameter characterizes the statistics as the Weibull distribution (Figs. 2 and 4). Details of fitting parameters for different degrees are given in Tables 1 and 2. Note that the KS test accepts the normal distribution as well for  $p_{in} = 0$ . This is due to a close resemblance of the Weibull distribution with the normal distribution for a particular parameter regime [14]. Furthermore,  $R_{min}$  lying in the left tail of the triangular shape distribution of eigenvalues of an ensemble of SF networks (at  $p_{in} = 0$ ) is known to follow a power law behaviour [15] which is bounded due to the finite size effect. This combination characterizes  $R_{min}$  for  $p_{in} = 0$  and consequently  $R_{max}$  for  $p_{in} = 1$  as the Weibull distribution. For the intermediate  $p_{in}$  values, except at  $p_{in} = 0.5$ , certain  $p_{in}$  values can be modeled using the GEV statistics but without any consistent behavior.

At  $p_{in} = 0.5$ , the statistics can be modeled using GEV statistics and exact form of the distribution depends on the average degree and the network size. At this  $p_{in}$  value, a transition is observed from the Weibull to the Fréchet via the Gumbel distribution as denseness ( $\langle k \rangle$ ) of the network increases ( Figs. 2 and 4). For a small network size, the reason behind this transition can be explained in terms of  $S_{max}$  behaviour of the network.  $S_{max}$  represents the maximum value of the column sum of a particular coupling matrix in a network ensemble. Figure 3 (a) displays distribution of  $S_{max}$  over an ensemble of the network coupling matrices for different values of the average degree. As reported in [16], a high value of  $S_{max}$  leads to the Fréchet distribution.  $S_{max}$  has the largest value for  $\langle k \rangle = 6$ , as shown in Fig.3. The value of  $S_{max}$  decreases for the lower average degrees displaying Gumbel (for  $\langle k \rangle = 4$ ) and Weibull (for  $\langle k \rangle = 2$ ) statistics. We further observe an interesting convergence behaviour of the shape parameter  $\xi$  characterizing the GEV distribution for  $R_{max}$  as we increase the size of the network, keeping average degree of the network fixed. We address the size impact on  $R_{max}$  distribution in a separate section.

Next, we compare the extreme value statistics of the isolated SF network with that of the ER networks. For  $p_{in} = 0.5$ , the ER networks exhibit the Weibull statistics for  $\langle k \rangle = 6$  and show a transition to the Fréchet via the Gumbel distribution as the average degrees increases [17]. Again, the behaviour of  $S_{max}$  can be used to explain appearance of the Weibull (for ER networks) and the Fréchet (for SF network) distribution at  $p_{in} = 0.5$  for a fixed network size and the average degree. Due to the hub-like structure, SF networks have a much higher largest degree than that of the corresponding ER networks. The high degree nodes of SF network ensures that  $S_{max}$  for an ER network will be much lesser than that of the SF network with the same average degree [Fig.3 (b)]. The high  $S_{max}$  values shift the shape parameter towards a more positive

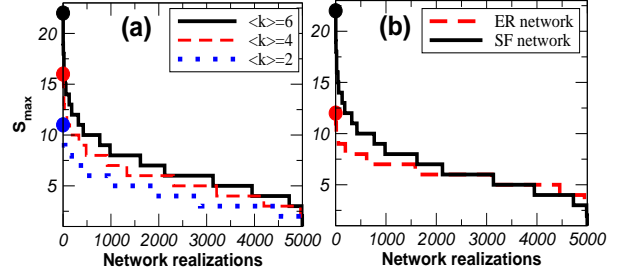


Fig. 3: (Color online)  $S_{max}$  as a function of random realizations of the matrices. (a)  $S_{max}$  for SF networks with  $\langle k \rangle = 6$  (black solid line),  $\langle k \rangle = 4$  (red dashed line),  $\langle k \rangle = 2$  (blue dotted line). (b)  $S_{max}$  for average degree  $\langle k \rangle = 6$  for SF (black solid line) and ER (red dashed line) networks. Other parameters are  $N=100$  and  $p_{in} = 0.5$ . The graph is plotted such that network with the highest  $S_{max}$  comes first.

value yielding the Fréchet distribution [16].

*Impact of Network Size on GEV statistics:* Our investigations demonstrate a profound impact of directionality on  $R_{max}$  distribution as the network size is increased. For  $p_{in} = 0$ , i.e. without any inhibition in the network coupling matrix, the shape parameter exhibits a very small variation in its value as a function of network size. The distribution can be modeled using the Weibull statistics even for the large network size irrespective of the average degree of the network. As inclusion of the inhibitory coupling resulting in negative entries in the network coupling matrices introduces a change in the behaviour of the shape parameter. For  $p_{in} = 0.5$ , there is an increment in the shape parameter as compared to that of  $p_{in} = 0$  for different average degrees and for a fixed network size. Moreover, we find that the shape parameter converges towards zero yielding the Gumbel distribution. For  $N = 100$ , the shape parameter  $\xi$  for the SF network displays Weibull (For  $\langle k \rangle = 2$ ), Gumbel (For  $\langle k \rangle = 4$ ) and Fréchet (for  $\langle k \rangle = 6, 8, 10, 20$ ) distribution. As we increase the network size, the value of shape parameter decreases for denser networks and takes a value which shows either the Gumbel distribution ((For example,  $\xi = 0.025$  for  $\langle k \rangle = 6$ )) or a close resemblance with the Gumbel distribution (For example,  $\xi = 0.059$  for  $\langle k \rangle = 10$ ) (Fig. 6). Further, for sparser networks, there is an increment in the shape parameter as a function of network size leading to transition from the Weibull to the Gumbel distribution for larger networks. We find that irrespective of the nature of  $R_{max}$  distribution portrayed initially for small networks, increment in the network size leads to fluctuations in  $R_{max}$  at same scale. These results can be interpreted in terms of the stability of a system. Tail behaviour of the Fréchet and the Gumbel distributions being power law and exponential decay, respectively indicate that for the Gumbel distribution higher values of  $R_{max}$  is less probable. Hence large systems displaying Gumbel distribution is more stable than the correspond-

Table 1: Estimated parameters of KS test for fitting of GEV and normal distributions of  $R_{max}$  for different inhibitory inclusion probability ( $p_{in}$ ) of SF network over 5000 population. Other parameters are network size  $N = 100$  and average degree  $\langle k \rangle = 6$ .

$p_{in}$	$\xi$ of GEV	$\sigma$ of GEV	$\mu$ of GEV	p-value of KS test for GEV	$\mu$ of Normal	$\sigma$ of Normal	p-value of KS test for Normal
0.0	-0.2065	0.1758	10.1715	0.0125	10.2413	0.1790	0.1158
0.2	-0.3580	1.4003	6.4256	0.0191	6.8590	1.3539	0.0000
0.4	0.0663	0.8681	3.3062	0.0000	3.8690	1.1556	0.0000
0.47	0.1586	0.5245	2.8023	0.1285	3.1987	0.8179	0.0000
0.49	0.0937	0.4955	2.7889	0.4406	3.1253	0.7177	0.0000
0.50	0.0597	0.4807	2.7905	0.9688	3.0983	0.6684	0.0000

Table 2: Estimated parameters of KS test for fitting GEV and normal distributions of  $R_{max}$  for different inhibitory inclusion probability ( $p_{in}$ ) of SF network over 5000 population. Other parameters are network size  $N = 100$  and average degree  $\langle k \rangle = 4$ .

$p_{in}$	$\xi$ of GEV	$\sigma$ of GEV	$\mu$ of GEV	p-value of KS test for GEV	$\mu$ of Normal	$\sigma$ of Normal	p-value of KS test for Normal
0.0	-0.2219	0.2109	7.302	0.1519	7.3843	0.2151	0.7846
0.2	-0.3038	0.9238	4.8985	0.0161	5.2143	0.909	0.0005
0.4	-0.0071	0.6743	2.8222	0.0002	3.211	0.8428	0.0000
0.46	0.0617	0.4901	2.4659	0.3708	2.7812	0.6710	0.0000
0.48	0.0309	0.4590	2.4275	0.5492	2.7077	0.6074	0.0000
0.50	-0.0049	0.4333	2.4219	0.8788	2.6700	0.5521	0.0000

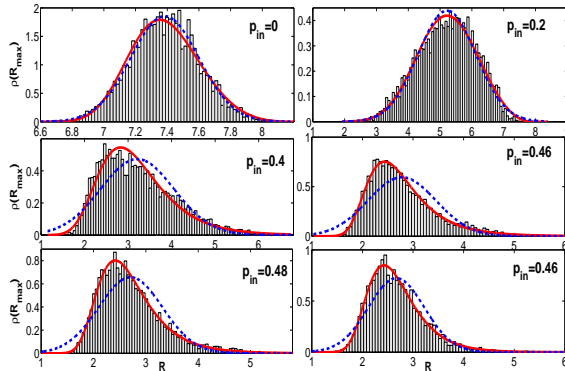


Fig. 4: (Color online) Distribution of  $R_{max}$  for 5000 population of SF networks with average degree  $\langle k \rangle = 4$  for various values of inhibitory probability ( $p_{in}$ ). Histogram is fitted with normal (blue dotted line) and GEV (red solid line) distributions.

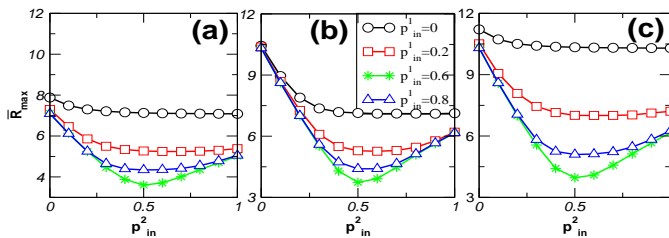


Fig. 5: (Color online) Average  $R_{max}$  as a function of inhibition probability ( $P_{in}^2$ ). (a)ER-ER network of  $\langle k \rangle = 6$ , (b)ER-SF, and (c)SF-SF network where ( $\bar{R}_{max}$ ) for various inhibition probability for layer 1,  $P_{in}^1 = 0(\circ)$ ,  $0.2(\square)$ ,  $0.6(*)$ ,  $0.8(\triangle)$ . Size of the network  $N = 100$ .

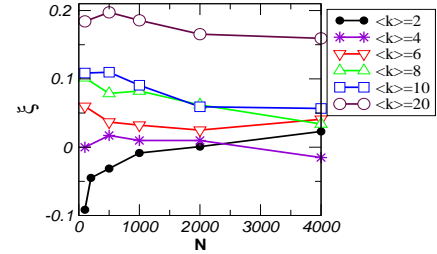


Fig. 6: (Color online) Shape parameter of GEV distribution as a function of network size ( $N$ ) for SF networks with average degree  $\langle k \rangle = 2(\bullet)$ ,  $\langle k \rangle = 4(*)$ ,  $\langle k \rangle = 6(\nabla)$ ,  $\langle k \rangle = 8(\triangle)$ ,  $\langle k \rangle = 10(\square)$ ,  $\langle k \rangle = 20(\circ)$  for  $p_{in} = 0.5$ .

ing systems with smaller network size which shows Fréchet statistics. Introduction of negative entries in network coupling matrices leads to a more stable system for the larger size and for the denser networks.

*Transition to Gumbel and Fréchet for multiplex network:* Next, we turn our attention to investigate the impact of multiplexing on  $R_{max}$  behaviour. It turns out that in absence of any inhibition, multiplexing of a network with another network leads to an enhancement in the  $R_{max}$  depending upon the architecture of the network it is multiplexing with and the impact of multiplexing is governed by the layer having the largest degree. A multiplexing of ER network with the SF network leads to an enhancement in  $R_{max}$  value of the entire network as compared to the isolated network, with the value of  $R_{max}$  lying approximately close to that of the isolated SF network as multiplexing increases degree of all the nodes in SF net-



work by one only and hence, there is no profound impact on  $R_{max}$  (Fig. 5). As inhibitory nodes are introduced with probability  $p_{in}$ , there is a trade off between the inhibition and the multiplexing, resulting in a quite interesting behaviour of  $R_{max}$ . Since, inhibition leads to make a matrix more anti-symmetric, the contribution from the layer having inhibition keeps on reducing as inhibition in that layer increases towards  $p_{in} = 0.5$  and impact of the layer having only symmetric coupling primarily governs  $R_{max}$ .

When ER network is multiplexed with another ER network, inhibition in only one layer does not have any potential impact on  $R_{max}$  as it is governed by the another ER layer where inhibition is not present. However, for the ER-SF multiplex networks, behaviour of  $R_{max}$  strongly depends on the layer in which inhibition is introduced. For instance, if inhibition is introduced in the ER layer,  $R_{max}$  remains un-affected as it is governed by the SF layer and since no inhibition is present in SF, leading to no visible impact on  $R_{max}$ . Value of  $R_{max}$  of the entire network, at no inhibition, is high due to this SF layer only. However, if inhibition is introduced in the SF layer, which leads to anti-symmetry in SF layer, that results in the decrease in the value of  $R_{max}$  of the entire network, finally reaching to a stable state when zero contribution comes from the SF layer (at  $p_{in} = 0.5$ ) and  $R_{max}$  of the entire network settles down to  $R_{max}$  of the isolated ER network. Thereafter, increasing inhibition in the SF layer does not have much impact as contribution to  $R_{max}$  comes from the ER layer as  $R_{max}$  of the isolated network for  $p_{in} > 0.5$  is always lower than that of the isolated ER network for  $p_{in} = 0$  (Figs. 1 and 5).

When both the layers are represented by SF networks, change in the  $R_{max}$  follows the similar behaviour as described for the ER-ER networks, as both the layers contribute equally to  $R_{max}$  and an inhibition in one layer makes another layer governing the  $R_{max}$  behaviour.

So far, we have discussed inhibition in only one layer of the multiplex network. As soon as we introduce inhibition in both the layers,  $R_{max}$  starts exhibiting several interesting phenomena which depends on trade-off of not only inhibition and multiplexing but trade-off between the inhibition in both the layers as well. Until inhibition in one layer (say 2), which has the larger highest degree, is lower than the inhibition in other layer,  $R_{max}$  keeps getting governed by the former and as soon as  $p_{in}$  in layer 2 gets larger than  $p_{in}$  of the layer 1,  $R_{max}$  starts getting governed by the layer 1. Interestingly, minima in  $R_{max}$  value, which a multiplex network can attain, is decided by the average degree of the multiplex network. The minima is reached, when inhibition in the SF layer becomes 0.5, as for this value maximum anti-symmetry is possible and consequently at this point  $R_{max}$  attains the minimum value governed by the average degree.

Next, we investigate fluctuations in the  $R_{max}$  behaviour for multiplex networks having inhibitory nodes in both or

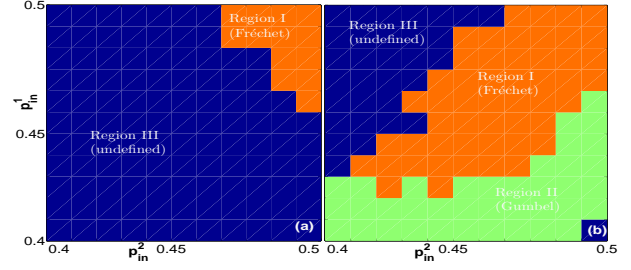


Fig. 7: (Color online) Phase diagram depicting shape parameter  $\xi$  for accepted GEV distribution in  $p_{in}^{(1)} - p_{in}^{(2)}$  plane for (a) SF-SF and (b) ER-SF multiplex network. Regions I and II correspond to the Fréchet and the Gumbel distribution, respectively. Region III stands for undefined distributions.  $N = 100$  in each layer and average is taken over 5000 random realizations of the network.

one of the layers. We present results for several possible combinations (ER-ER, ER-SF, SF-SF) and show that multiplexing with the similar type of the network does not lead to any significant impact on the  $R_{max}$  fluctuations, whereas multiplexing with a layer having a different network architecture has the following impact. The  $R_{max}$  statistics is again led by the layer having the largest degree as also found for the average  $R_{max}$  behaviour. But what is surprising that the range of  $p_{in}$  for which the fluctuations can be modeled by the GEV statistics gets enhanced due to the multiplexing. For example, if both the layers of the multiplex network are represented by the ER networks, the  $R_{max}$  statistics can be characterized by the Weibull distribution for  $p_{in}^{(1)} = 0.5$  and  $0.4 \leq p_{in}^{(2)} \leq 0.5$  [18] as also reflected for the isolated case [17]. If both the layers are represented by the SF networks,  $R_{max}$  of an ensemble of the multiplex network does not fit with any distribution (Region III, Fig.7(b)) except at a narrow range around  $p_{in}^{(1)} = p_{in}^{(2)} = 0.5$  (Region I, Fig.7(a)) where it shows GEV statistics. Type of the statistics in this region depends upon the average degree of the network. As average degree increases, one gets a transition from the Weibull to the Fréchet distribution.

While  $R_{max}$  of the multiplex network having layers being represented with a similar architecture does not manifest any change as compared to that of the isolated case, an interesting phenomenon is observed for multiplex networks with different network topologies representing different layers. An ER network multiplexed with a SF network results in an increment in the shape parameter. Note that, the isolated ER and isolated SF exhibits the Weibull and the Fréchet statistics respectively. The multiplex ER-SF network exhibits a very different  $R_{max}$  fluctuation behaviour due to the interplay between the multiplexity and the inhibition. For instance, Fig.7 depicts  $R_{max}$  fluctuation statistics of ER-SF network for average degree  $\langle k \rangle = 6$ . The  $R_{max}$  distribution can be characterized by the Gumbel (Region II, Fig.7(b)) as well as

the Fréchet (Region I, Fig.7(b)) distribution in the range  $0.4 \leq p_{in}^{(1)} = p_{in}^{(2)} \leq 0.5$ . There is a transition from the Weibull to the Gumbel statistics for the parameter range  $0.4 \leq p_{in}^{(1)} \leq 0.5, p_{in}^{(2)} = 0.42$ , arising due to the interplay between ER and SF layers. Because of the large degree nodes in the SF networks, we observe the Fréchet region for parameter range  $0.41 \leq p_{in}^{(1)} \leq 0.5$  and  $0.42 \leq p_{in}^{(2)} \leq 0.5$ . Although, SF layer governs the  $R_{max}$  fluctuation behaviour, we find the Gumbel regime for a parameter range which reflects that the incremental behaviour of the shape parameter is greatly enhanced due to the multiplexing of a network with a different network architecture. This observation further indicates a reduction in the stability of a network as well as change in the dynamical behaviour of a network when it is multiplexed with another network having a different topology.

*Conclusion:* To summarize, we have investigated impact of inhibitory and excitatory coupling on  $R_{max}$  statistics for an ensemble of isolated and multiplex networks. While, the isolated SF networks for high connection density exhibit a transition from the Weibull to Fréchet distribution as a function of inhibition probability  $p_{in}$ , the SF networks with lower connection density does not show such transition for denser as well as sparser networks. In terms of stability of the corresponding network, this indicates that a higher values of  $R_{max}$  indicating a higher probability to become un-stable for denser network with fixed network size. Moreover, multiplexity is shown to have a strong influence on the stability of a network. The  $R_{max}$  behaviour deviates drastically when it is multiplexed with a network of different network topology. Our study demonstrates that the multiplex network becomes more unstable than the isolated network even when directionality was introduced in a single layer of the multiplex network.

Furthermore, we demonstrate a surprising effect of the network size on the  $R_{max}$  behaviour of the network at  $p_{in} = 0.5$ . For denser networks, the  $R_{max}$  distribution is shown to have a transition from the Fréchet to the Gumbel distribution for larger networks. Whereas for sparser networks  $R_{max}$  exhibits a transition from Weibull to Gumbel as a function of network size indicating lesser stability. This indicates that the stability of the system increases for large dense network where as it decreases for large sparser networks. Fluctuations in the largest eigenvalue converges to a particular distribution for larger networks irrespective of the connection density.

The extreme value statistics tools has found its applicability in a wide range of real world systems. The largest mass distribution in mass transport network exhibits the Weibull, Gumbel or Fréchet statistics depending on the critical mass [19]. Further, the level density of non-interacting bosons is shown to follow GEV distributions [20]. Our work extends the application of extreme value statistics to the multiplex networks. The investigation

presented here can be used to understand stability and dynamical behaviour of complex systems having multiplex architecture and can be extended further to understand effect of different layers interacting with inhibitory connections on collective properties of real world systems having inherent multilayer architecture.

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